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A UNIFIED STEADY-STATE ANALYSIS FOR CONTROLLED MARKOV DRIFT PROCESSES IN INVENTORY, QUEUEING AND REPLACEMENT PROBLEMS

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A unified steady-state analysis for controlled Markov drift processes in inventory, queueing and replacement problems *)

by

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ABSTRACT

This paper presents a simple and unified approach for the steady state analysis of a class of controlled Markov drift processes involving compound Poisson processes and arising in various applications areas as inventory production, queueing and replacement systems. In these problems the drift parameters of the underlying stochastic processes can be controlled and the control rule considered is characterized by two switch-over levels.

KEY WORDS & PHRASES: *Markov drift processes, controllable drift parameters, switch-over levels, steady-state analysis, applications, inventory, queueing and replacement systems.*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In this paper we shall present a simple and unified approach for the steady-state analysis of a class of controlled Markov drift processes involving compound Poisson processes and arising in various application areas as inventory, production, queueing and replacement systems. We consider the following three different problems.

a. *A finite capacity queueing-storage problem with variable input and output rates.*

Consider a storage system (e.g. a dam) whose inventory level can be controlled by varying the input and output rates. The system has a finite storage capacity K. Assume that at each epoch the system can be classified to be into one of two possible phases $i = 1,2$ where at any point of time the system can be switched from one phase to the other without loss of time. If the system is in phase i then at epochs generated by a Poisson process with rate λ_i a positive random amount S_i of material is added to the content of the system where an overflow occurs for any excess of the capacity level K. Between the epochs of content additions we have in phase 1 that the inventory level increases linearly at rate $\sigma_1 > 0$ when the system is not full and in phase 2 that the inventory level decreases linearly at rate $\sigma_2 > 0$ when the inventory is positive.

The following cost structure is imposed on the model. There are operating costs at rate $h_i(x)$ when the system is in phase i and the inventory is x where the functions $h_1(x)$ and $h_2(x)$ are bounded and have only a finite number of discontinuities in $0 \le x \le K$. Further, an overflow cost of $p_i(y)$ is incurred when an overflow of an amount y occurs in phase i and a fixed switch-over cost of R is incurred when the system is switched from one phase to the other.

The rule for controlling the inventory is characterized by two switch-over levels y_1 and y_2 with $0 \le y_2 < y_1 \le K$. This rule prescribes to switch the system from phase 1 to phase 2 only when the inventory reaches or exceeds the level y_1 and prescribes to switch the system from phase 2 to phase 1 only when the inventory is less than or equal to y_2.

b. *A finite capacity production-inventory problem with variable demand and production rates.*

Consider a production-system in which a single product is produced and kept in stock where the demand and production rates can be controlled. There is a finite storage capacity K. The system can be classified to be into one of two possible phases $i = 1,2$ where at any point of time the system can be switched from one phase to the other without loss of time. If the system is in phase i then at epochs generated by a Poisson process with rate λ_i a demand for a positive random amount S_i of the product occurs where any demand in excess of stock is lost (the backorder case can be handled quite similarly). Between demand epochs we have in phase 1 that the inventory level decreases linearly at rate $\sigma_1 > 0$ when the inventory is positive and in phase 2 that the inventory level increases linearly at rate $\sigma_2 > 0$ when the system is not full.

A same cost structure as in the first problem is imposed on the model. The rule for controlling the inventory prescribes to switch the system from phase 1 to phase 2 only when the inventory reaches or falls below y_1 and prescribes to switch the system from phase 2 to phase 1 only when the inventory is larger than or equal to y_2 where y_1 and y_2 are given control parameters with $0 \leq y_1 < y_2 \leq K$.

c. *A replacement problem under additive damage with variable damaging rates.*

Consider a machine or production system which is subject to shocks occurring at epochs generated by a Poisson process and causing random amounts of damage where between stocks the cumulative damage gradually changes. The damages accumulate additively and can be controlled. A machine whose cumulative damage exceeds a given level M is always replaced by a new machine having no damage. The system can be classified to be into one of two possible phases $i = 1,2$ where at any point of time the system can be switched from one phase to the other without loss of time. If the machine is in phase i then at epochs generated by a Poisson process with rate λ_i shocks occur and cause a positive random amount S_i of damage. Between shocks we have in phase 1 that the cumulative damage increases linearly at rate $\sigma_1 > 0$ (e.g. by aging) and in phase 2 that the cumulative damage decreases linearly at rate $\sigma_2 > 0$ (e.g. by repair). The replacement of a machine in phase i with cumulative damage x requires an amount of time $\tau_i(x)$ where $E\tau_i(x)$ is bounded in x. A new machine is initially in phase 1 and has damage zero.

The following cost structure is imposed on the model. There is an operating cost at rate $h_i(x)$ when the machine is in phase i and has cumulative damage x where $h_i(x)$ is bounded and has only a finite number of discontinuities in x. A nonnegative replacement cost of $R_i(x)$ is incurred when a machine in phase i and with cumulative damage x is replaced. Finally there is a fixed cost of R_1 (R_2) for switching the machine from phase 1 (2) to phase 2 (1).

The rule for controlling the cumulative damage prescribes to switch the machine from phase 1 to phase 2 only when the cumulative damage lies between y_1 and M including the endpoints and prescribes to switch the machine from phase 2 to phase 1 only when the cumulative damage is less than or equal to y_2 where y_1 and y_2 are given control parameters with $0 \le y_2 < y_1 \le M$. The machine is replaced by a new one when the cumulative damage exceeds the given level M.

In each of the above control problems we are interested in a formula for the *long-run average expected costs per unit time* under the given control rule. By an appropriate choice of the cost parameters, we may obtain from this formula various *operating characteristics* for the system like the *stationary probability distribution of the state of the system, the average number of switch-overs and overflows per unit time, etc., cf.* [12].

The above three problems are different but closely related to each other. Roughly speaking, the first two problems are in essence equivalent and the third problem mainly differs from the first one by the fact that in the third problem a "discontinuous" jump of the state occurs by a replacement. In this paper we shall derive for the third problem a formula for the long-run average expected costs per unit time. The approach to be used applies equally well to the other two problems. A slightly more complex but essentially the same approach was applied in [11]-[12] to a somewhat different version of the first problem with $\sigma_1 < 0$, cf. also [4], [5], [9] and [10]. We note that the above problems include as special cases a variety of problems previously studied in the literature, e.g. cf. [1], [3], [7] and [8].

In section 2 we first discuss an integro-differential equation which plays an essential role in the analysis and next in section 3 we present the derivation of a formula for the long-run average costs.

2. AN INTEGRO-DIFFERENTIAL EQUATION.

Let $a(x)$ be a given bounded function having only a finite number of discontinuities in an interval $a \le x \le b$ and let α be a given nonzero constant. Further let F be a given probability distribution function with $F(0) = 0$ and finite first moment μ. Suppose that the unknown function $u(x)$ is continuous on $[a,b]$ and for all except countably many $x \in (a,b)$ satisfies the integro-differential equation

$$\frac{du(x)}{dx} = a(x) + \alpha\{-u(x) + \int_0^{b-x} u(x+y)dF(y)\} \tag{1}$$

where some boundary condition for $u(x)$ is known. Throughout this paper any interval of integration is closed unless stated otherwise. This integro-differential equation can be reduced to a standard renewal equation so that the solution of (1) can be given. Therefore we first note that, by partial integration (cf. p. 77 in [2]),

$$\frac{d}{dx}\int_0^{b-x} u(x+y)\,\{1-F(y)\}dy = -u(x) + \int_0^{b-x} u(x+y)dF(y) \tag{2}$$

for any x such that F is continuous at $b-x$. Hence, by (1), (2), the continuity of u and the fact that F has only a countable number of discontinuities, we find

$$u(x) = A(x) + \alpha \int_0^{b-x} u(x+y)\,\{1-F(y)\}dy \qquad \text{for } a \le x \le b \tag{3}$$

where

$$A(x) = \gamma + \int_0^{x} a(y)\,dy \qquad \text{for } a \le x \le b.$$

for some constant γ to be determined by the given boundary condition for u. The equation (3) is a (delayed) renewal equation when $|\alpha|\mu \le 1$. However, in many situations of interest the case $|\alpha|\mu > 1$ will occur. To solve (3) simultaneously for both cases, we follow [6] and define the distribution function G and the number δ by

$$G(x) = 0 \quad \text{for } x < 0,\ G(x) = |\alpha| \int_0^{x} \{1-F(y)\}dy \quad \text{for } x \ge 0 \tag{4}$$

and

$$\int_0^{\infty} e^{-\delta y}dG(y) = 1. \tag{5}$$

Since G is non-negative and non-increasing with $G(\infty) = |\alpha|\mu < \infty$, it follows that δ is uniquely determined by (5) and that the function H defined by

$$H(x) = 0 \text{ for } x < 0 \text{ and } H(x) = \int_0^x e^{-\delta y} dG(y) \quad \text{for } x \geq 0 \tag{6}$$

is a probability distribution function concentrated on $(0,\infty)$. Denote by $G^{(n)}$ and $H^{(n)}$ the n-fold convolution of G and H with itself respectively, then it is readily verified that

$$H^{(2)}(x) = \int_0^x e^{-\delta y} dG^{(2)}(y) \quad \text{for } x \geq 0, \tag{7}$$

and so, by $H^{(2)}(\infty) = 1$,

$$\int_0^\infty e^{-\delta y} dG^{(2)}(y) = 1. \tag{8}$$

Further, define for the probability distribution functions H and $H^{(2)}$ the corresponding renewal functions M and $\bar{M}$ by

$$M(x) = \sum_{n=1}^{\infty} H^{(n)}(x) \text{ and } \bar{M}(x) = \sum_{n=1}^{\infty} H^{(2n)}(x) \quad \text{for all } x.$$

We are now in a position to solve (1). We distinguish between two cases.

Case (i). $\alpha > 0$. Then, by (4), we can write (3) as

$$u(x) = A(x) + \int_0^{b-x} u(x+y)dG(y) \quad \text{for } a \leq x \leq b,$$

or equivalently, by (6), (cf. also p. 77 in [2] or pp. 362-363 in [6]),

$$e^{\delta x}u(x) = e^{\delta x}A(x) + \int_0^{b-x} e^{\delta(x+y)}u(x+y)dH(y) \quad \text{for } a \leq x \leq b.$$

This equation is a standard renewal equation for the function $u^*(x) = e^{\delta x}u(x)$. The solution of this renewal equation gives

$$u(x) = A(x) + \int_0^{b-x} e^{\delta y}A(x+y)\, dM(y) \quad \text{for } a \leq x \leq b. \tag{9}$$

Case (ii). $\alpha < 0$. Then, by (4), we can write (3) as

$$u(x) = A(x) - \int_0^{b-x} u(x+y)dG(y) \qquad \text{for } a \le x \le b$$

and so, by substitution,

$$u(x) = B(x) + \int_0^{b-x} u(x+y)\, dG^{(2)}(y) \qquad \text{for } a \le x \le b,$$

where

$$B(x) = A(x) - \int_0^{b-x} A(x+y)dG(y) \qquad \text{for } a \le x \le b.$$

In the same way as in case (i), we now find, by using (7)-(8),

$$u(x) = B(x) + \int_0^{b-x} e^{\delta y} B(x+y)\, d\bar{M}(y) \quad \text{for } a \le x \le b. \tag{10}$$

In the special case of $F(x) = 1-e^{-\eta x}$ the quantities δ, M and $\bar{M}$ can be explicitly given by

$$\delta = |\alpha|-\eta, \; \frac{dM(y)}{dy} = |\alpha|, \; \frac{d\bar{M}(y)}{dy} = |\alpha| e^{-|\alpha|y} \{e^{|\alpha|y} - e^{-|\alpha|y}\}/2.$$

In general M and $\bar{M}$ cannot be explicitly given although useful approximations are known (cf. [6]). However, much numerical work needs still to be done for computing $u(x)$ from (1).

3. THE DERIVATION OF THE AVERAGE COSTS.

Denote by $Z(t)$ the total costs incurred up to time t for $t \ge 0$. Defining a cycle as the time interval between two successive epochs at which a new machine starts operating after a replacement, we have by a standard result in the theory of regenerative processes that

$$\lim_{t\to\infty} \frac{EZ(t)}{t} = \frac{\text{total expected costs incurred during one cycle}}{\text{expected length of one cycle}} .$$

In fact we also have that, with probability 1, $Z(t)/t$ converges to the right side of this relation as $t \to \infty$.

To determine the above ratio, define for $0 \le x \le y_1$,

$K(x)$ = total expected cost incurred until the next epoch at which a new machine starts operating after a replacement given that at epoch 0 the system is in phase 1 and has cumulative damage x.

$T(x)$ = expected time until the next epoch at which a new machine starts operating after a replacement given that at epoch 0 the system is in phase 1 and has cumulative damage x.

Then

$$\lim_{t\to\infty} \frac{EZ(t)}{t} = \frac{K(0)}{T(0)} . \tag{11}$$

It suffices to determine $K(x)$ since an expression for $T(x)$ follows from the one for $K(x)$ by taking $h_i(x) = 1$, $R_i(x) = E\tau_i(x)$ and $R_1 = R_2 = 0$. To determine $K(x)$, we define in its turn the quantities

$k_1(x)$ = total expected operating and replacement costs incurred until the first epoch at which the cumulative damage reaches or exceeds the level y_1 given that at epoch 0 the system is in phase 1 and has cumulative damage x $(0 \le x \le y_1)$.

$p_1(x,v)$ = the probability that at the first epoch at which the cumulative damage reaches or exceeds the level y_1 the value of the cumulative damage is less than or equal to v given that at epoch 0 the system is in phase 1 and has cumulative damage x $(0 \le x \le y_1,\ v \ge y_1)$.

$k_2(x)$ = total expected operating and replacement costs incurred until the first epoch at which the cumulative damage either decreases to the level y_2 or exceeds the replacement level M given that at epoch 0 the system is in phase 2 and has cumulative damage x $(y_2 \le x \le M)$.

$p_2(x)$ = probability that the cumulative damage first decreases to y_2 before it exceeds M given that at epoch 0 the system is in phase 2 and has cumulative damage x $(y_2 \le x \le M)$.

Note that $k_1(y_1) = k_2(y_2) = 0$ and $p_1(y_1,v) = p_2(y_2) = 1$ for any $v \ge y_1$.

Clearly, for any $0 \leq x \leq y_1$,

$$K(x) = k_1(x) + \int_{y_1}^{M} p_1(x,dv)\,[R_1 + k_2(v) + p_2(v)\,\{R_2 + K(y_2)\}]. \tag{12}$$

Once we have determined the quantities k_1, p_1, k_2 and p_2, then we get $K(y_2)$ and hence $K(x)$ for $0 \leq x \leq y_1$ from (12). To determine these quantities, we first observe that the functions $k_1(x)$, $k_2(x)$, $p_2(x)$ and $p_1(x,v)$ for any v are continuous in x. We shall now derive an integro-differential equation for $k_1(x)$. Considering $k_1(x-\Delta x)$ for Δx small and using standard arguments, we find for all except countably many $x \in (0,y_1)$

$$k_1(x-\Delta x) = h_1(x)\,\frac{\Delta x}{\sigma_1} + \lambda_1\,\frac{\Delta x}{\sigma_1}\,\{\int_{(M-x,\infty)} R_1(x+y)\,dF_1(y) + \int_0^{y_1-x} k_1(x+y)\,dF_1(y)\} + (1 - \lambda_1\,\frac{\Delta x}{\sigma_1})k_1(x),$$

where F_i denotes the probability distribution function of the random variable S_i representing the damage caused by a shock occurring in phase i. Hence for all except countably many $x \in (0,y_1)$ we have

$$\frac{dk_1(x)}{dx} = \frac{-h_1(x)}{\sigma_1} - \frac{\lambda_1}{\sigma_1}\int_{(M-x,\infty)} R_1(x+y)\,dF_1(y) + - \frac{\lambda_1}{\sigma_1}\,\{-k_1(x) + \int_0^{y_1-x} k_1(x+y)\,dF_1(y)\} \tag{13}$$

with the boundary condition

$$k_1(y_1) = 0. \tag{14}$$

This integro-differential equation is of the type (1). In the same way, we find for any $v \geq y_1$ that for all except countably many $x \in (0,y_1)$

$$\frac{\partial p_1(x,v)}{\partial x} = -\frac{\lambda_1}{\sigma_1}\,\{F_1(v-x) - F_1(y_1-x)\} - \frac{\lambda_1}{\sigma_1}\,\{-p_1(x,v) + \int_0^{y_1-x} p_1(x+y,v)\,dF_1(y)\} \tag{15}$$

with the boundary condition

$$p_1(y_1,v) = 1. \tag{16}$$

Similarly, for all except countably many $x \in (y_2, M)$,

$$\frac{dk_2(x)}{dx} = \frac{h_2(x)}{\sigma_2} + \frac{\lambda_2}{\sigma_2} \int_{(M-x,\infty)} R_2(x+y)\, dF_2(y) + $$

$$+ \frac{\lambda_2}{\sigma_2} \{-k_2(x) + \int_0^{M-x} k_2(x+y)\, dF_2(y)\} \tag{17}$$

with the boundary condition $k_2(y_2) = 0$. Finally, for all except countably many $x \in (y_2, M)$,

$$\frac{dp_2(x)}{dx} = \frac{\lambda_2}{\sigma_2} \{1 - F_2(M-x)\} + \frac{\lambda_2}{\sigma_2} \{-p_2(x) + \int_0^{M-x} p_2(x+y) dF_2(y)\} \tag{18}$$

with the boundary condition $p_2(y_2) = 1$.

The integro-differential equations (13), (15), (17) and (18) are of type (1) and consequently they can be solved in terms of the renewal functions M and $\bar{M}$. For the special case where F_1 and F_2 are exponential distribution functions the solutions can be explicitly given.

We remark that the above analysis is quite flexible and may be routinely extended when switch-over times are involved, a given machine can only be switched once from phase 1 to phase 2, etc. We may also consider the case where between shock epochs the cumulative damage changes at a general rate $\sigma_i(x)$ satisfying some regularity conditions. Then we get instead of (1) an integro-differential equation of the type

$$\sigma(x) \frac{du(x)}{dx} = a(x) + \alpha\{-u(x) + \int_0^{b-x} u(x+y)\, dF(y)\}.$$

In general such an integro-differential equation can be only solved numerically where for an exponential distribution function F this equation can be reduced to a first-order linear differential equation in $du(x)/dx$ by differentiation.

We finally remark that, by choosing $R_1(x) = R_2(x) = h_2(x) = R_1 = R_2 = 0$, $h_1(x) = 1$ for $x \le z$ and $h_1(x) = 0$ for $x > z$ with z is a given number, the long-run average expected cost per unit time gives the long-run expected fraction of time during which the system is operating in phase 1 with a cumulative damage of at most z. In this way we get the stationary probability distribution of the state of the system.

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